

Functions of Mittag-Leffler and Fox: The Pathway Model to Tsallis Statistics and Beck-Cohen Superstatistics

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Abstract

In reaction rate theory, in production-destruction type models and in reaction-diffusion problems when the total derivatives are replaced by fractional derivatives the solutions are obtained in terms of Mittag-Leffler functions and their generalizations. When fractional calculus enters into the picture the solutions of these problems, usually available in terms of generalized hypergeometric functions, switch to Mittag-Leffler functions and their generalizations into Wright functions and subsequently into Fox functions. In this paper, connections are established among generalized Mittag-Leffler functions, Mathai's pathway model, Tsallis statistics, Beck-Cohen superstatistics, and among corresponding entropic measures. The Mittag-Leffler function, for large values of the parameter, approaches a power-law. For values of the parameter close to zero, the Mittag-Leffler function behaves like a stretched exponential. The Mittag-Leffler function is a generalization of the exponential function and represents a deviation from the exponential paradigm whenever it shows up in solution of physical problems. The paper elucidates the relation between analytic representations of the q -exponential function that is fundamental to Tsallis statistics, Mittag-Leffler, Wright, and Fox functions, respectively, utilizing Mellin-Barnes integral representations.

1. Introduction

Fundamental laws of physics are written as equations for the time evolution of a quantity $x(t)$,

$$\frac{dx(t)}{dt} = Ax(t), \quad (1.1)$$

where if A is limited to a linear operator we have Maxwell's equation or Schroedinger equation, or it could be Newton's law of motion or Einstein's equations for geodesics if A may also be a nonlinear operator [8]. When A is linear then the mathematical solution is

$$x(t) = x_0 e^{-At} \quad (1.2)$$

where x_0 is the initial value at $t = 0$. Taking the simplest case of (1.1), as used in reaction rate theory ([8],[4]), if the number density at time t of the i -th particle is $N_i(t)$ and if the number of particles produced or the production rate is proportional to $N_i(t)$ then the reaction equation is

$$\frac{dN(t)}{dt} = k_1 N(t), \quad k_1 > 0$$

deleting i for convenience. If the decay rate is also proportional to $N(t)$ then the corresponding equation is

$$\frac{dN(t)}{dt} = -k_2 N(t), \quad k_2 > 0.$$

Then the residual effect in a production-destruction mechanism is of the form

$$\frac{dN(t)}{dt} = -cN(t), \quad c > 0 \Rightarrow N(t) - N_0 = -c \int N(t) dt \quad (1.3)$$

if the destruction rate dominates so that the production-destruction model is a decaying model. Such production-destruction models abound in scientific disciplines. If the total integral or the total derivative in (1.3) is replaced by a fractional integral then we have

$$N(t) - N_0 = -c^\nu {}_0D_t^{-\nu} N(t), \quad (1.4)$$

where ${}_0D_t^{-\nu}$ is the Riemann-Liouville fractional integral operator ([5],[7]) defined by

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du \quad (1.5)$$

for $\Re(\nu) > 0$ where $\Re(\cdot)$ denotes the real part of (\cdot) , c is replaced by c^ν for convenience, and then the solution of (1.4) goes into the category of a Mittag-Leffler function [4], namely,

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c^k t^k)^\nu}{\Gamma(1+k\nu)} = N_0 E_\nu[-(ct)^\nu] \quad (1.6)$$

where $E_\nu(\cdot)$ is the Mittag-Leffler function. The generalized Mittag-Leffler function is defined as

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k! \Gamma(\beta + \alpha k)}, \quad \Re(\alpha) > 0, \Re(\beta) > 0, \quad (1.7)$$

and when $\Gamma(\gamma)$ is defined, it has the following Mellin-Barnes representation

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)} \sum_{k=0}^{\infty} \frac{\Gamma(\gamma + k)}{k! \Gamma(\beta + \alpha k)} \quad (1.8)$$

$$= \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s) \Gamma(\gamma - s)}{\Gamma(\beta - \alpha s)} (-z)^{-s} ds \quad (1.9)$$

for $0 < c < \Re(\gamma)$, $\Re(\gamma) > 0$, $i = \sqrt{-1}$. Some special cases of the generalized Mittag-Leffler function are the following:

$$E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta + k\alpha)} z^k \quad (1.10)$$

$$E_{\alpha,1}^1 = E_{\alpha,1}(z) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + k\alpha)} \quad (1.11)$$

and when $\alpha = 1$ we have

$$E_\alpha(z) = E_1(z) = e^z. \quad (1.12)$$

Thus the Mittag-Leffler function can be looked upon as an extension of the exponential function. The Mellin-Barnes representation in (1.9) is a special case of the Mellin-Barnes representation of the Wright's function [3] which is defined as

$$\begin{aligned} {}_p\psi_q(z) &= {}_p\psi_q \left[z \middle| \begin{matrix} (a_j, \alpha_j), j=1, \dots, p \\ (b_j, \beta_j), j=1, \dots, q \end{matrix} \right] \\ &= \sum_{k=0}^{\infty} \frac{\prod_{j=1}^p \Gamma(a_j + \alpha_j k)}{\prod_{j=1}^q \Gamma(b_j + \beta_j k)} \frac{z^k}{k!} \end{aligned} \quad (1.13)$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) \left\{ \frac{\prod_{j=1}^p \Gamma(a_j - \alpha_j s)}{\prod_{j=1}^q \Gamma(b_j - \beta_j s)} \right\} (-z)^{-s} ds \quad (1.14)$$

where $0 < c < \min_{1 \leq j \leq p} \Re\left(\frac{a_j}{\alpha_j}\right)$ with $a_j, j = 1, \dots, p$ and $b_j, j = 1, \dots, q$ being complex quantities and $\alpha_j > 0, j = 1, \dots, p$ and $\beta_j > 0, j = 1, \dots, q$ being real quantities. Observe from (1.14) that Wright's function is a special case of the H-function [7] and the H-function is defined as the following Mellin-Barnes integral

$$H_{p,q}^{m,n}(z) = H_{p,q}^{m,n} \left[z \middle| \begin{matrix} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{matrix} \right] = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(s) z^{-s} ds \quad (1.15)$$

where

$$\phi(s) = \frac{\left\{ \prod_{j=1}^m \Gamma(b_j + \beta_j s) \right\} \left\{ \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s) \right\}}{\left\{ \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s) \right\} \left\{ \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s) \right\}}, \quad (1.16)$$

for $\max_{1 \leq j \leq m} \frac{\Re(-b_j)}{\beta_j} < c < \min_{1 \leq j \leq n} \frac{\Re(1-a_j)}{\alpha_j}$ where $a_j, j = 1, \dots, p$ and $b_j, j = 1, \dots, q$ are complex quantities, $\alpha_j > 0, j = 1, \dots, p$ and $\beta_j > 0, j = 1, \dots, q$ are real quantities. Existence conditions and various contours may be seen from books on the H-function, for example, ([5],[7]). Equations (1.8), (1.14), and (1.15) establish the relations between Mittag-Leffler, Wright, and Fox functions to be used in the following for q-exponential function in Tsallis statistics, Beck-Cohen superstatistics, and their link to Mathai's pathway model.

1.1. Extension of the reaction rate model and Mittag-Leffler function

The reaction rate model in (1.3) can be extended in various directions ([8],[4]). For example, if N_0 is replaced by $N_0 f(t)$ where $f(t)$ is a general integrable function on the finite interval $[0, b]$ then it is easy to see that for the solution of the equation

$$N(t) - N_0 f(t) = -c^\nu {}_0D_t^{-\nu} N(t) \quad (1.17)$$

there holds the formula

$$N(t) = cN_0 \int_0^t H_{1,2}^{1,1} \left[c^\nu (t-x)^\nu \middle| \begin{matrix} (-\frac{1}{\nu}, 1) \\ (-\frac{1}{\nu}, 1), (0, \nu) \end{matrix} \right] f(x) dx. \quad (1.18)$$

Some special cases are the following: Let $\nu > 0, \rho > 0, c > 0$. Then for the solution of the fractional equation

$$N(t) - N_0 t^{\rho-1} = -c^\nu {}_0D_t^{-\nu} N(t) \quad (1.19)$$

there holds the formula

$$N(t) = N_0 \Gamma(\rho) t^{\rho-1} E_{\nu, \rho}(-ct)^\nu. \quad (1.20)$$

Let $c > 0, \nu > 0, \mu > 0$. Then for the solution of the fractional equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}^{\gamma} [-(ct)^{\nu}] = -c^{\nu} {}_0D_t^{-\nu} N(t) \quad (1.21)$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} E_{\nu,\mu}^{\gamma+1} [-(ct)^{\nu}]. \quad (1.22)$$

1.2. Fractional partial differential equations and Mittag-Leffler functions

In a series of papers, see for example, ([2],[4]) it is illustrated that the solutions of certain fractional partial differential equations, resulting from fractional diffusion problems, are available in terms of Mittag-Leffler functions. For example, consider the equation

$${}_0D_t^{\nu} N(x, t) - \frac{t^{-\nu}}{\Gamma(1-\nu)} \delta(x) = -c^{\nu} \frac{\partial^2}{\partial x^2} N(x, t) \quad (1.23)$$

with initial conditions

$${}_0D_t^{\nu-k} N(x, t)|_{t=0} = 0, k = 1, \dots, n$$

where $n = [\Re(\nu)] + 1$, c^{ν} is the diffusion constant, $\delta(x)$ is the Dirac's delta function and $[\Re(\nu)]$ is the integer part of $\Re(\nu)$. The solution of (1.23), by taking Laplace transform with respect to t and Fourier transform with respect to x and then inverting, can be shown to be of the form

$$N(x, t) = \frac{1}{(4\pi c^{\nu} t^{\nu})^{\frac{1}{2}}} H_{1,2}^{2,0} \left[\frac{|x|^2}{4c^{\nu} t^{\nu}} \middle| \begin{matrix} (1-\frac{\nu}{2}, \nu) \\ (0,1), (\frac{1}{2}, 1) \end{matrix} \right] \quad (1.24)$$

which in special cases reduce to Mittag-Leffler functions.

This paper is organized as follows: Section 2 gives the classical special function technique of eliminating the upper or lower parameters from a general hypergeometric series. Section 3 establishes the pathways of going from Mittag-Leffler function to pathway models to Tsallis statistics and Beck-Cohen superstatistics through the parameter elimination technique. Section 4 gives representations of Mittag-Leffler functions and pathway model in terms of H-functions and then gives a pathway to go from a Mittag-Leffler function to the pathway model through H-function.

2. A classical special function technique

The well established technique of eliminating a numerator or denominator parameter from a general hypergeometric function in the classical theory of special functions, is the following: For convenience, we will illustrate it on a confluent hypergeometric function.

$${}_1F_1(a; b; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}, (a)_m = a(a+1)\dots(a+m-1), a \neq 0, (a)_0 = 1. \quad (2.1)$$

Observe that

$$\begin{aligned} \frac{(a)_k}{a^k} &= \frac{a}{a} \frac{(a+1)}{a} \dots \frac{(a+m-1)}{a} \\ &= 1(1 + \frac{1}{a})(1 + \frac{2}{a}) \dots (1 + \frac{m-1}{a}) \rightarrow 1 \text{ as } a \rightarrow \infty \end{aligned} \quad (2.2)$$

for all finite k . Similarly

$$\frac{b^k}{(b)_k} \rightarrow 1 \text{ when } b \rightarrow \infty. \quad (2.3)$$

Therefore

$$\lim_{a \rightarrow \infty} {}_1F_1(a; b; \frac{z}{a}) = {}_0F_1(-; b; z) \quad (2.4)$$

which is a Bessel function. Thus we can go from a confluent hypergeometric function to a Bessel function through this process. Similarly

$$\lim_{b \rightarrow \infty} {}_1F_1(a; b; bz) = {}_1F_0(a; -; z) = (1 - z)^{-a}, \quad |z| < 1. \quad (2.5)$$

Thus, from a confluent hypergeometric function we can go to a binomial function. Further,

$$\lim_{b \rightarrow \infty} {}_0F_1(-; b; bz) = {}_0F_0(-; -; z) = e^z \quad (2.6)$$

and

$$\lim_{a \rightarrow \infty} {}_1F_0(a; -; \frac{z}{a}) = {}_0F_0(-; -; z) = e^z. \quad (2.7)$$

Thus, we can go from a Bessel function as well as from a binomial function to an exponential function. These two results can be stated in a slightly different form as follows:

$$\lim_{q \rightarrow 1} {}_0F_1(-; \frac{1}{q-1}; -\frac{z}{q-1}) = e^{-z} \quad (2.8)$$

and

$$\lim_{q \rightarrow 1} {}_1F_0(\frac{1}{q-1}; -; -(q-1)z) = \lim_{q \rightarrow 1} [1 + (q-1)z]^{-\frac{1}{q-1}} = e^{-z}. \quad (2.9)$$

Equation (2.9) is the starting point of Tsallis statistics [9]. The left side in (2.9) is the q -exponential function of Tsallis, namely,

$$\begin{aligned} [1 + (q-1)z]^{-\frac{1}{q-1}} &= \exp_q(-z) \\ &= e^{-z} \text{ when } q \rightarrow 1. \end{aligned} \quad (2.10)$$

We consider a more general form of (2.9) given by

$$\begin{aligned} \lim_{q \rightarrow 1} c|x|^\gamma {}_1F_0(\frac{\eta}{q-1}; -; -a(q-1)|x|^\delta) &= \lim_{q \rightarrow 1} c|x|^\gamma [1 + a(q-1)|x|^\delta]^{-\frac{\eta}{q-1}} \\ &= c|x|^\gamma e^{-a\eta|x|^\delta}, \quad a > 0, \eta > 0, -\infty < x < \infty. \end{aligned} \quad (2.11)$$

Thus, we obtain Mathai's pathway model [6], namely,

$$f(x) = c|x|^\gamma [1 + a(q-1)|x|^\delta]^{-\frac{\eta}{q-1}}, \quad a > 0, \eta > 0 \quad (2.12)$$

where c is the normalizing constant. If $\eta = 1, \gamma = 0, a = 1, \delta = 1, x > 0$ then (2.12) gives Tsallis statistics [9]. For $q > 1, a = 1, \eta = 1, x > 0$ in (2.12) we get Beck-Cohen superstatistics [1].

It may be observed that (2.7), which is the binomial form or ${}_1F_0$ going to exponential, is exploited to produce the pathway model, Tsallis statistics and Beck-Cohen superstatistics whereas (2.6), which is the Bessel function form going to exponential is not yet exploited. This produces a rich variety of applicable functions as will be shown in a forthcoming paper.

3. Connection of Mittag-Leffler function to the pathway model

In order to see the connection, let us recall the Mellin-Barnes representation of the generalized Mittag-Leffler function.

$$\Gamma(\beta)E_{\alpha,\beta}^{\gamma}(z^{\delta}) = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\gamma-s)\Gamma(s)}{\Gamma(\beta-\alpha s)} (-z^{\delta})^{-s} ds \quad (3.1)$$

for $\Re(\gamma) > 0, i = \sqrt{-1}, \Re(\beta) > 0$. Let us examine the situation when $|\beta| \rightarrow \infty$. We will state two basic results as lemmas.

Lemma 3.1 *When $|\beta| \rightarrow \infty$ and $\alpha > 0$ is finite then*

$$\lim_{|\beta| \rightarrow \infty} \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha s)} [-b(z\beta^{\frac{\alpha}{s}})^{\delta}]^{-s} = (-bz^{\delta})^{-s}. \quad (3.2)$$

Proof. The Stirling's formula is given by

$$\Gamma(z + a) \approx \sqrt{2\pi} z^{z+a-\frac{1}{2}} e^{-z}, \text{ for } |z| \rightarrow \infty \quad (3.3)$$

and a is a bounded quantity. Now, applying Stirling's formula we have

$$\begin{aligned} \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha s)} [-b(x\beta^{\frac{\alpha}{s}})^{\delta}]^{-s} &\approx \frac{\sqrt{2\pi}\beta^{\beta-\frac{1}{2}}e^{-\beta}}{\sqrt{2\pi}\beta^{\beta-\frac{1}{2}-\alpha s}e^{-\beta}} [-b(x\beta^{\frac{\alpha}{s}})^{\delta}]^{-s} \\ &= \beta^{\alpha s} (-bz^{\delta})^{-s} \beta^{-\alpha s} = (-bz^{\delta})^{-s}. \end{aligned} \quad (3.4)$$

Lemma 3.2. *For $\Re(\gamma) > 0, \Re(\beta) > 0$,*

$$\lim_{|\beta| \rightarrow \infty} \Gamma(\beta)E_{\alpha,\beta}^{\gamma}(-bz^{\delta}) = \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(\gamma-s)\Gamma(s) [bz^{\delta}]^{-s} ds \quad (3.5)$$

$$= \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!} (-bz^{\delta})^k = [1 + bz^{\delta}]^{-\gamma} \text{ for } |bz^{\delta}| < 1. \quad (3.6)$$

Proof. After applying Lemma 3.1 we can evaluate the contour integral in (3.5) by using the residue theorem at the poles of $\Gamma(s)$ to obtain the series form in (3.6). For $|bz^{\delta}| > 1$ we obtain the series form as analytic continuation or by evaluating the integral in (3.5) as the sum of the residues at the poles of $\Gamma(\gamma-s)$. This will be equal to the following:

$$\lim_{|\beta| \rightarrow \infty} \Gamma(\beta)E_{\alpha,\beta}^{\gamma}(-bz^{\delta}) = (bz^{\delta})^{-\gamma} [1 + (bz^{\delta})^{-1}]^{-\gamma} \quad (3.7)$$

for $|bz^{\delta}| > 1$ which will be of the same form as in (3.6). Replacing b by $a(q-1)$ and γ by $\frac{\eta}{q-1}$ we have the following:

$$\begin{aligned} \lim_{|\beta| \rightarrow \infty} c|z|^{\gamma} \Gamma(\beta)E_{\alpha,\beta}^{\eta/(q-1)}(-a(q-1)|z|^{\delta}) \\ = c|z|^{\gamma} [1 + a(q-1)|z|^{\delta}]^{-\frac{\eta}{q-1}}. \end{aligned} \quad (3.8)$$

Observe that (3.8) is nothing but Mathai's pathway model [6] for $a > 0, \delta > 0, \eta > 0$ with c being the normalizing constant, for both $q > 1$ and $q < 1$. Then when $q \rightarrow 1$ (3.8) will reduce to the exponential form. That is,

$$\lim_{q \rightarrow 1} \lim_{|\beta| \rightarrow \infty} c|z|^{\gamma} \Gamma(\beta)E_{\alpha,\beta}^{\eta/(q-1)}[-a(q-1)|z|^{\delta}] = c|z|^{\gamma} e^{-a\eta|z|^{\delta}}. \quad (3.9)$$

Note from (3.8) that when $q > 1$ the functional form in (3.8) remains the same whether $\delta > 0$ or $\delta < 0$ with $-\infty < z < \infty$. When $q < 1$ the support in (3.8) will be different for $\delta > 0$ and $\delta < 0$ if $f(x)$ is to remain as a statistical density.

For $\gamma = 0, a = 1, \delta = 1, \eta = 1, z > 0$ we have Tsallis statistics [9] coming from (3.8) for both the cases $q > 1$ and $q < 1$. For $q > 1, a = 1, \delta = 1, \eta = 1, z > 0$ we have Beck-Cohen superstatistics [1] coming from (3.8). Observe that the limiting process from (3.5) to (3.6) holds for $\gamma = 1$ or $\gamma = 1$ and $\alpha = 1$ also. Thus the special cases of Mittag-Leffler function are also covered.

Thus, through β a pathway is created to go from a generalized Mittag-Leffler function to Mathai's pathway model and then to Tsallis statistics and Beck-Cohen superstatistics. If β is real then as β becomes larger and larger then

$$c|z|^\gamma \Gamma(\beta) E_{\alpha, \beta}^{\eta/(q-1)}(-a(q-1)|z|^\delta)$$

goes closer and closer to the pathway model in (3.8). In other words, a pathway is created through β to go from a Mittag-Leffler function to the pathway model to Tsallis statistics and Beck-Cohen superstatistics. Thus for large real value of β or for large value of $|\beta|$,

$$c|z|^\gamma \Gamma(\beta) E_{\alpha, \beta}^{\eta/(q-1)}[-a(q-1)|z|^\delta] \approx c|z|^\gamma [1 + a(q-1)|z|^\delta]^{-\frac{\eta}{q-1}}. \quad (3.10)$$

4. Connections through the H-function

Recalling the generalized Mittag-Leffler function from (3.1) and representing it in terms of a H-function we have the following:

$$\Gamma(\beta) E_{\alpha, \beta}^\gamma (z\beta^{\frac{\alpha}{q}})^\delta = \frac{\Gamma(\beta)}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\gamma-s)}{\Gamma(\beta-\alpha s)} [-(z\beta^{\frac{\alpha}{q}})^\delta]^{-s} ds \quad (4.1)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[-(z\beta^{\frac{\alpha}{q}})^\delta \middle|_{(0,1),(1-\beta,\alpha)}^{(1-\gamma,1)} \right]. \quad (4.2)$$

From the limiting process discussed in (3.2) we have the following result:

Lemma 4.1 For $\Re(\beta) > 0, \Re(\gamma) > 0$,

$$\begin{aligned} \lim_{|\beta| \rightarrow \infty} \frac{\Gamma(\beta)}{\Gamma(\gamma)} H_{1,2}^{1,1} \left[-(z\beta^{\frac{\alpha}{q}})^\delta \middle|_{(0,1),(1-\beta,\alpha)}^{(1-\gamma,1)} \right] \\ = \frac{1}{\Gamma(\gamma)} H_{1,1}^{1,1} \left[-z^\delta \middle|_{(0,1)}^{(1-\gamma,1)} \right] \end{aligned} \quad (4.3)$$

$$= \frac{1}{\Gamma(\gamma)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\Gamma(\gamma-s) (-z^\delta)^{-s} ds \quad (4.4)$$

$$= [1 - z^\delta]^{-\gamma}. \quad (4.5)$$

Therefore we have the following theorem:

Theorem 4.1 For $\Re(\beta) > 0, \Re(\gamma) > 0, x > 0, a > 0, q > 1, c > 0$

$$\begin{aligned} \lim_{|\beta| \rightarrow \infty} c \frac{\gamma(\beta)}{\Gamma\left(\frac{\eta}{q-1}\right)} x^\gamma E_{\alpha, \beta}^{\eta/(q-1)} [-a(q-1)\beta^{\frac{\alpha}{q}} x]^\delta \\ = cx^\gamma [1 + a(q-1)x^\delta]^{-\frac{\eta}{q-1}}. \end{aligned} \quad (4.6)$$

Observe that the right side in (4.6) is the pathway model [6] for $x > 0$ from where one has Tsallis statistics, Beck-Cohen superstatistics and power law where the constant c can act as the normalizing constant to create a statistical density in the right side of (4.6).

We will write the right side in (4.6) as a H-function and then establish a connection between Mittag-Leffler function and the pathway model through the H-function. To this end, the practical procedure is to look at the Mellin transform of the right side of (4.6) and then write it as a H-function by taking the inverse Mellin transform. The Mellin transform of the right side of (4.6), denoted by $M_f(s)$, is given by the following:

$$\begin{aligned} M_f(s) &= \int_0^\infty c x^{\gamma+s-1} [1 + a(q-1)x^\delta]^{-\frac{\eta}{q-1}} dx, a > 0, q > 1, \eta > 0, \delta > 0 \\ &= \frac{c}{\delta} \frac{\Gamma\left(\frac{\gamma+s}{\delta}\right)}{[a(q-1)]^{\frac{\gamma+s}{\delta}}} \frac{\Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+s}{\delta}\right)}{\Gamma\left(\frac{\eta}{q-1}\right)} \end{aligned} \quad (4.7)$$

for $\Re\left(\frac{\gamma+s}{\delta}\right) > 0, \Re\left(\frac{\eta}{q-1} - \frac{\gamma+s}{\delta}\right) > 0$. Note that if c is the normalizing constant for the density $f(x)$ then by putting $s = 1$ the right side of (4.7) must be 1. Therefore,

$$M_f(s) = \frac{[a(q-1)]^{\frac{1}{\delta}}}{\Gamma\left(\frac{\gamma+1}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta}\right)} \Gamma\left(\frac{\gamma+s}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+s}{\delta}\right) [a(q-1)]^{-\frac{s}{\delta}}. \quad (4.9)$$

Hence the right side of (4.6) is available as the inverse Mellin transform of (4.9). That is,

$$f(x) = cx^\gamma [1 + a(q-1)x^\delta]^{-\frac{\eta}{q-1}}, \quad (4.10)$$

for $a > 0, \eta > 0, q > 1, \delta > 0, x > 0$

$$\begin{aligned} &= \frac{[a(q-1)]^{\frac{1}{\delta}}}{\Gamma\left(\frac{\gamma+1}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta}\right)} \\ &\times \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma\left(\frac{\gamma+s}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+s}{\delta}\right) [x(a(q-1))^{\frac{1}{\delta}}]^{-s} ds \end{aligned} \quad (4.11)$$

$$= \frac{[a(q-1)]^{\frac{1}{\delta}}}{\Gamma\left(\frac{\gamma+1}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta}\right)} H_{1,1}^{1,1} \left[x(a(q-1))^{\frac{1}{\delta}} \left| \begin{matrix} (1 - \frac{\eta}{q-1} + \frac{\gamma}{\delta}, \frac{1}{\delta}) \\ (\frac{\gamma}{\delta}, \frac{1}{\delta}) \end{matrix} \right. \right] \quad (4.12)$$

$$= \lim_{|\beta| \rightarrow \infty} c \frac{\Gamma(\beta)}{\Gamma\left(\frac{\eta}{q-1}\right)} x^\gamma E_{\alpha,\beta}^{\eta/(q-1)} [-x(a(q-1))^{\frac{1}{\delta}} \beta^{\frac{\alpha}{\delta}}]^\delta \quad (4.13)$$

where

$$c = \frac{\delta [a(q-1)]^{\frac{\gamma+1}{\delta}} \Gamma\left(\frac{\eta}{q-1}\right)}{\Gamma\left(\frac{\gamma+1}{\delta}\right) \Gamma\left(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta}\right)} \quad (4.14)$$

for $q > 1, \delta > 0, \eta > 0, a > 0, \Re(\gamma+1) > 0, \Re\left(\frac{\eta}{q-1} - \frac{\gamma+1}{\delta}\right) > 0$.

Remark 4.1. From (4.13) and (4.10) it can be noted that as the parameter β becomes larger and larger the Mittag-Leffler function in (4.13) comes closer and closer to the pathway model and eventually in the limiting situation both become identical. In a physical situation, if (4.10) represents the stable situation then the unstable neighborhoods are given by (4.13). When $q \rightarrow 1$ then (4.10) goes to the exponential form. Thus if the exponential form is the stable situation then the pathway model of (4.1) itself models the unstable neighborhoods. This unstable neighborhood is farther extended by the Mittag-Leffler form in (4.13).

Remark 4.2. Observe that the functional form in (4.10) remains the same whether $q > 1$ or $q < 1$. But for $q < 1$ or when $q \rightarrow 1$ the normalizing constant c will be different.

Remark 4.3. When dealing with problems such as reaction-diffusion situations or a general production-destruction model one goes to fractional differential equations to get a better picture of the solution. Then we usually end up in Mittag-Leffler functions and their generalizations into Wright's function or Fox function. Comparison of (4.13) and (4.10) reveals that as β gets larger and larger the effect of fractional derivative becomes less and less and finally when $|\beta| \rightarrow \infty$ the effect of taking fractional derivatives, instead of total derivatives, gets nullified.

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